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Free Vibrations of an Orthotropic Thin Cylindrical Shell on a Pasternak Foundation

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The basic objective of the present study is to investigate the influences of foundation parameters, orthotropy of materials represented by elastic moduli ratio, axial wave parameter, and circumferential wave number on three eigenfrequencies. Dynamic equilibrium equations and frequency equation are developed, and the eigenfrequencies are calculated. It is found that the foundation modulus primarily affects the radial mode eigenfrequency and has no effect on torsional and longitudinal modes, whereas the shear modulus does have influence on radial as well as tangential modes of vibration, though the effect on radial mode frequency is more dominating. The elastic moduli ratio chiefly affects the radial mode in a manner similar to that of a foundation modulus. When the shell rests on a Pasternak foundation, the lowest eigenfrequency vs circumferential wave-number characteristics do not have the well-known dip, and the minimum frequency occurs when the circumferential wave number is zero for all values of axial wave parameter.

Nomenclature

C_1, C_2, C_3	=	vibration amplitude constants in axial, tangential, and radial directions, respectively
E_1, E_2	=	Young's moduli in x and ϕ directions, respectively
G	=	shear modulus of shear layer of Pasternak foundation
$\bar{G}$	=	nondimensional shear modulus, $G(1-\nu_{12}\nu_{21}))/E_1 h$
G_{12}	=	modulus of rigidity associated with principal material directions
$\overline{G_{12}}$	=	nondimensional modulus of rigidity associated with principal material directions, $G_{12}(1-\nu_{12}\nu_{21}))/E_1$
H	=	shell thickness
K	=	foundation modulus
k	=	$h^2/12R^2$
L	=	length of the cylindrical shell
M	=	number of axial half-waves
N	=	number of circumferential waves
R	=	radius of the middle surface of the shell
T	=	time coordinate
u, v, w	=	displacement components of the middle surface in axial, circumferential, and radial directions, respectively (w-positive inward)
α	=	E_1/E_2
λ	=	axial wave parameters, $\lambda^* R = m\pi R/L$
λ^*	=	wave parameter, $m\pi/L$
$\bar{\mu}$	=	nondimensional foundation parameter, $KR^2(1-\nu_{12}\nu_{21}))/E_1 h$
ν_{12}, ν_{21}	=	Poisson's ratios in x and ϕ directions, respectively
ρ	=	mass density of structural material
χ_1, χ_2	=	curvature
χ_{12}	=	twist
Ω	=	dimensionless frequency parameter
ω	=	angular frequency

Introduction

THE underground and buried oil and gas pipelines are continuously in contact with elastic soil on the outer surface and compressible fluids on the inner surface. Similarly, undersea oil and gas pipelines and tubes of the heat exchangers remain in contact with fluids on both sides. Such pipelines and tubes can also be considered as thin cylindrical shells on an elastic foundation. Rockets and missiles filled with solid and liquid fuels are another example of cylindrical shells on an elastic foundation. Shallow shells supported on soft and light filaments in space vehicles and boilers and storage tanks on floor grid work in ships are some of the other instances of shells on elastic foundation. Shells of field guns can also be considered as cylindrical shells on an elastic foundation. Most earthen soils can be appropriately represented by a mathematical model from Pasternak, whereas sandy soils and liquids can be represented by Winkler's model. Underground and undersea pipelines are often subjected to dynamic loads caused by seismic forces, sea waves, and nuclear explosions. Similarly, tube bundles of heat exchangers are subjected to flow-induced vibrations as well as vibrations emanating from attached pumps and compressors. One notices a great spurt in the wide and extensive applications of pipes made of composites during the past decade. Hence vibration analysis of orthotropic cylindrical shells resting on an elastic foundation is of paramount importance. Markus[1] has presented an excellent description of vibrations of isotropic and multilayered cylindrical shells. Paliwal and Pandey[2] studied free vibrations of an isotropic cylindrical shell on an elastic foundation employing bending theory. Recently Paliwal and Singh[3] presented the vibrational behavior of orthotropic shells on elastic foundations using membrane theory. However, the membrane theory does not take into account the change of the eigenfrequency caused by a change in wall thickness. It also does not interpret the effect of bending stiffness and rotational inertia. Herein, the authors have presented the free vibration analysis of orthotropic shells on a Pasternak foundation based on bending theory. The effect of axial wave parameter, circumferential wave number, and the foundation parameters on the vibration frequencies has been extensively studied. The lowest eigenfrequency, predominantly the radial vibrational mode being the most vital and sensitive, is investigated in more detail.

Analysis

Using bending theory, the following equations of equilibrium are obtained for a thin cylindrical shell resting on a Pasternak foundation:

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\cdot\frac{\partial N_{x\phi}}{\partial \phi} + p_x = 0 \tag{1a}$$

$$\frac{1}{R}\cdot\frac{\partial N_\phi}{\partial \phi} + \frac{\partial N_{x\phi}}{\partial x} - \frac{1}{R}\cdot\frac{\partial M_{x\phi}}{\partial x} - \frac{1}{R^2}\cdot\frac{\partial M_\phi}{\partial \phi} + p_y = 0 \tag{1b}$$

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$$\frac{\partial^2 M_x}{\partial x^2} + \frac{2}{R} \cdot \frac{\partial^2 M_{x\phi}}{\partial\phi\partial x} + \frac{1}{R^2} \cdot \frac{\partial^2 M_\phi}{\partial x^2} + \frac{N_\phi}{R} + p_z = 0 \quad \text{(1c)}$$

where

$$p_x = -\rho h \frac{\partial^2 u}{\partial t^2}, \qquad p_y = -\rho h \frac{\partial^2 v}{\partial t^2}, \qquad p_z = -\rho h \frac{\partial^2 w}{\partial t^2} - p$$

$$p = Kw - G\nabla^2 w = \text{Pasternak foundation reaction}$$

The following strain-displacement and stress-strain relations are employed to obtain the final governing equation:

$$\varepsilon_x = \frac{\partial u}{\partial x} - z\chi_1 \quad \text{(2a)}$$

$$\varepsilon_\phi = \frac{1}{R} \cdot \frac{\partial v}{\partial \phi} - \frac{w}{R} - z\chi_2 \quad \text{(2b)}$$

$$\gamma_{x\phi} = \frac{1}{R} \cdot \frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x} - 2z\chi_{12} \quad \text{(2c)}$$

$$N_x = \int_{-h/2}^{+h/2} \sigma_x \cdot \mathrm{d}z = \frac{E_1}{(1 - \nu_{12}\nu_{21})} \int_{-h_{1/2}}^{+h_{1/2}} (\varepsilon_x + \nu_{21}\varepsilon_\phi)\,\mathrm{d}z; \cdots \quad \text{(3a)}$$

$$M_x = \int_{-h/2}^{+h/2} \sigma_x \cdot z\,\mathrm{d}z = \frac{E_1}{(1 - \nu_{12}\nu_{21})} \int_{-h_{1/2}}^{+h_{1/2}} (\varepsilon_x + \nu_{21}\varepsilon_\phi) z\,\mathrm{d}z; \cdots \quad \text{(3b)}$$

The substitution of Eqs. (2) and (3) into Eq. (1) yields the following governing equations:

$$\frac{\partial^2 u}{\partial x^2} + \left[\frac{\nu_{21}}{R} + \frac{G_{12}(1 - \nu_{12}\nu_{21})}{E_1 R}\right]\frac{\partial^2 v}{\partial\phi\partial x} + \frac{G_{12}(1 - \nu_{12}\nu_{21})}{E_1 R^2} \cdot \frac{\partial^2 u}{\partial \phi^2} - \frac{\nu_{21}}{R} \cdot \frac{\partial w}{\partial x} = \frac{\rho(1 - \nu_{12}\nu_{21})}{E_1} \cdot \frac{\partial^2 u}{\partial t^2} \quad \text{(4a)}$$

$$\left[\frac{G_{12}(1 - \nu_{12}\nu_{21})}{E_2} + \frac{G_{12}h^2(1 - \nu_{12}\nu_{21})}{6E_2R^2}\right]\frac{\partial^2 v}{\partial x^2} + \left[\frac{1}{R^2} + \frac{1}{12}\left(\frac{h}{R^2}\right)^2\right]\frac{\partial^2 v}{\partial \phi^2} + \left[\frac{\nu_{12}}{R} + \frac{G_{12}(1 - \nu_{12}\nu_{21})}{E_2 R}\right]\frac{\partial^2 u}{\partial x \partial\phi} + \left[\frac{\nu_{12}}{12}\left(\frac{h}{R}\right)^2 + \frac{G_{12}h^2(1 - \nu_{12}\nu_{21})}{6E_2R^2}\right]\frac{\partial^3 w}{\partial x^2\partial\phi} + \frac{1}{12}\left(\frac{h}{R^2}\right)^2\frac{\partial^3 w}{\partial\phi^3} - \left(\frac{1}{R^2}\right)\frac{\partial w}{\partial \phi} = \frac{\rho(1 - \nu_{12}\nu_{21})}{E_2} \cdot \frac{\partial^2 v}{\partial t^2} \quad \text{(4b)}$$

$$\left(-\frac{E_1h^2}{12E_2}\right)\frac{\partial^4 w}{\partial x^4} - \left[\frac{E_1h^2\nu_{21}}{12R^2E_2} + \frac{G_{12}h^2(1 - \nu_{12}\nu_{21})}{3R^2E_2} + \frac{h^2\nu_{12}}{12R^2}\right]\frac{\partial^4 w}{\partial x^2\partial\phi^2} - \frac{1}{12}\left(\frac{h}{R^2}\right)^2\frac{\partial^4 w}{\partial\phi^4} - \left[\frac{E_1h^2\nu_{21}}{12E_2R^2} + \frac{G_{12}h^2(1 - \nu_{12}\nu_{21})}{3E_2R^2}\right]\frac{\partial^3 v}{\partial x^2\partial\phi} - \frac{1}{12}\left(\frac{h}{R^2}\right)^2\frac{\partial^3 v}{\partial\phi^3} + \left(\frac{1}{R^2}\right)\frac{\partial v}{\partial\phi} + \left(\frac{\nu_{12}}{R}\right)\frac{\partial u}{\partial x} - \left[\frac{1}{R^2} + \frac{K(1 - \nu_{12}\nu_{21})}{E_2 h}\right]w + \frac{G(1 - \nu_{12}\nu_{21})}{E_2 h}\left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{R^2} \cdot \frac{\partial^2 w}{\partial\phi^2}\right) = \frac{\rho(1 - \nu_{12}\nu_{21})}{E_2} \cdot \frac{\partial^2 w}{\partial t^2} \quad \text{(4c)}$$

Now the general solution of the preceding dynamic equilibrium equation with boundary conditions, $v = w = N_x = M_x = 0$ at points $x = 0$ and $x = L$ for simply supported shell, can be sought in the form

$$u = C_1 \cos\lambda^* x \cos n\phi \cos\omega t \quad \text{(5a)}$$

$$v = C_2 \sin\lambda^* x \sin n\phi \cos\omega t \quad \text{(5b)}$$

$$w = C_3 \sin\lambda^* x \cos n\phi \cos\omega t \quad \text{(5c)}$$

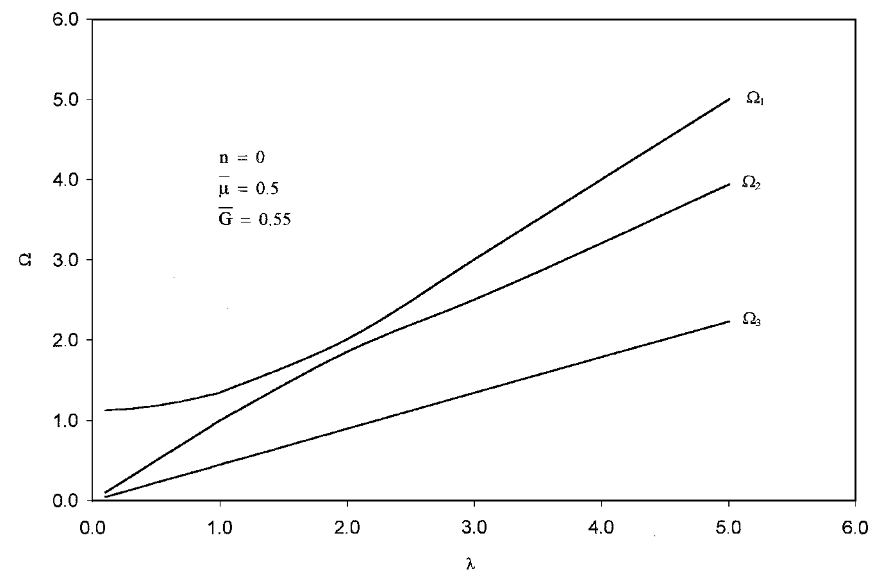

Fig. 1 Roots of the frequency equation vs axial wave parameter for shell on a Pasternak foundation ($n = 0$).

where

$$\lambda^* = \lambda/R = m\pi/L$$

where m is the number of half-waves in the longitudinal direction and n the number of half-waves in the circumferential direction.

Equations (4) can be rewritten as

$$\frac{\partial^2 u}{\partial x^2} + S_1 \cdot \frac{\partial^2 v}{\partial \phi \partial x} + S_2 \cdot \frac{\partial^2 u}{\partial \phi^2} - S_3 \frac{\partial w}{\partial x} = S_4 \cdot \frac{\partial^2 u}{\partial t^2} \tag{6a}$$

$$S_5 \frac{\partial^2 v}{\partial x^2} + S_6 \frac{\partial^2 v}{\partial \phi^2} + S_7 \frac{\partial^2 u}{\partial x \partial \phi} + S_8 \frac{\partial^3 w}{\partial \phi^3} + S_9 \frac{\partial^3 w}{\partial x^2 \partial \phi} - S_{10} \frac{\partial w}{\partial \phi} = S_{11} \frac{\partial^2 v}{\partial t^2} \tag{6b}$$

$$S_{12} \frac{\partial^4 w}{\partial x^4} - S_{13} \frac{\partial^4 w}{\partial x^2 \partial \phi^2} - S_{14} \frac{\partial^4 w}{\partial \phi^4} - S_{15} \frac{\partial^3 v}{\partial x^2 \partial \phi} - S_{16} \frac{\partial^3 v}{\partial \phi^3} + S_{17} \frac{\partial v}{\partial \phi} + S_{18} \frac{\partial u}{\partial x} - S_{19} w + S_{20} \frac{\partial^2 w}{\partial x^2} + S_{21} \frac{\partial^2 w}{\partial \phi^2} = S_{22} \frac{\partial^2 w}{\partial t^2} \tag{6c}$$

where S_1 to S_{22} are defined in the Appendix.

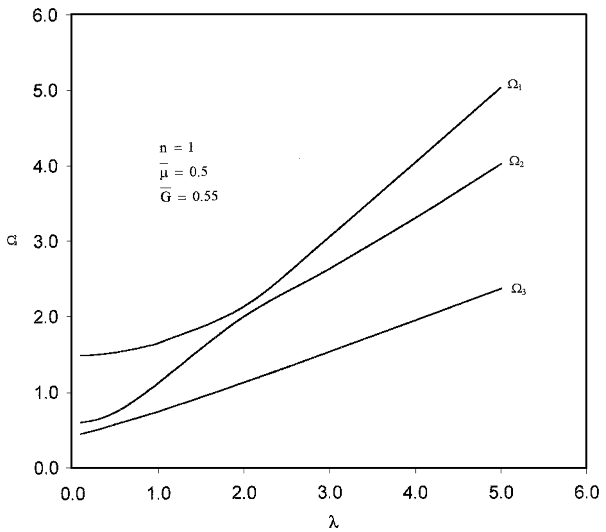

Fig. 2 Roots of the frequency equation vs axial wave parameter for shell on a Pasternak foundation (n = 1).

Substituting expressions (5) for displacement components u, v, w in Eqs. (6), we get

$$\left(\omega^2 S_4 - \lambda^{*2} - n^2 S_2\right) R^2 C_1 + \left(n\lambda^* S_1\right) R^2 C_2 + \left(-\lambda^* S_3\right) R^2 C_3 = 0 \tag{7a}$$

$$\left(n\lambda^* S_7\right) R^2 C_1 + \left(-\lambda^{*2} S_5 - n^2 S_6 + \omega^2 S_{11}\right) R^2 C_2 + \left(n^3 S_8 + n\lambda^{*2} S_9 + n S_{10}\right) R^2 C_3 = 0 \tag{7b}$$

$$\left(-\lambda^* S_{18}\right) R^2 C_1 + \left(n\lambda^{*2} S_{15} + n^3 S_{16} + n S_{17}\right) R^2 C_2 + \left(\omega^2 S_{22} + \lambda^{*4} S_{12} - n^2 \lambda^{*2} S_{13} - n^4 S_{14} - S_{19} - S_{20}\lambda^{*2} - n^2 S_{21}\right) R^2 C_3 = 0 \tag{7c}$$

For the nontrivial solution of Eqs. (7), the determinant of the coefficients of the preceding three simultaneous equations must be equal to zero, which can be expressed as

$$\begin{vmatrix} \Omega^2 - A_1 & A_2 & A_3 \\ A_4 & \alpha\Omega^2 - A_5 & A_6 \\ A_7 & A_8 & \alpha\Omega^2 - A_9 \end{vmatrix} = 0 \tag{8}$$

where A_1 to A_9 are defined in the Appendix

The expansion of the determinant (8) results in

$$\bar{\Omega}^3 + Q_1 \bar{\Omega}^2 + Q_2 \bar{\Omega} + Q_3 = 0 \tag{9}$$

where $\bar{\Omega} = \Omega^2$ and

$$Q_1 = -(1/\alpha)(\alpha A_1 + A_5 + A_9)$$

$$Q_2 = (1/\alpha^2)[\alpha(A_1 A_5 + A_1 A_9 - A_2 A_4 - A_3 A_7) + (A_5 A_9 - A_6 A_8)]$$

$$Q_3 = (1/\alpha^2)(A_3 A_5 A_7 + A_3 A_4 A_8 + A_2 A_4 A_9 + A_2 A_6 A_7 - A_1 A_5 A_9 + A_1 A_6 A_8)$$

The three roots of cubic equation (9) give three vibration frequencies, Ω_1, Ω_2, and Ω_3.

Discussion

Figure 1 shows that major changes occur in the shape of the frequency of parameter curves, when the cylindrical shell is in contact with a Pasternak foundation. Not only the torsional frequency parameter but even the radial frequency varies linearly with the longitudinal wave parameter λ. The three roots of frequency parameter

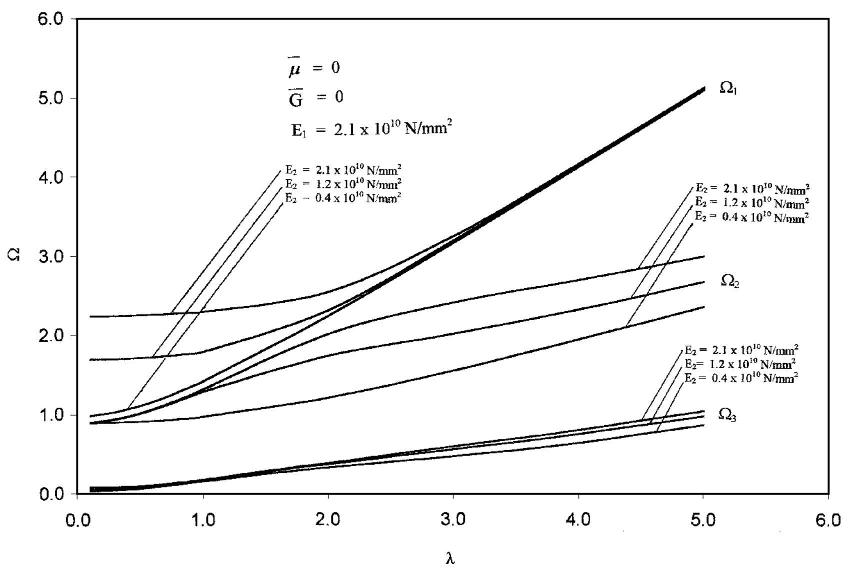

Fig. 3 Eigenfrequency vs axial wave parameter for various elastic moduli ratios (n = 2).

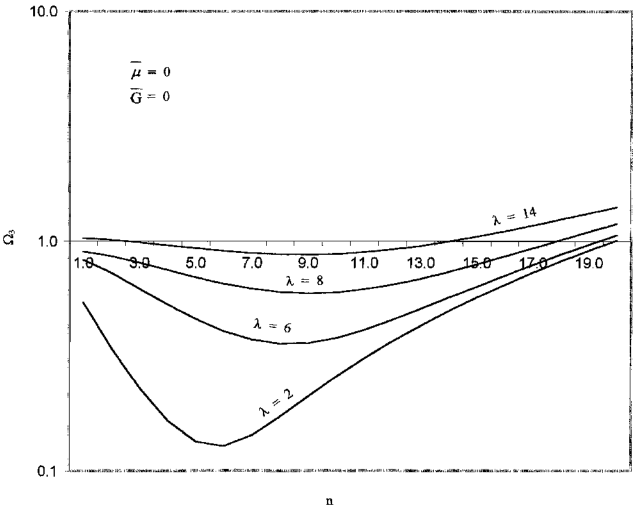

Fig. 4 Lowest eigenfrequency vs circumferential wave number for various axial wave parameters (without elastic foundation).

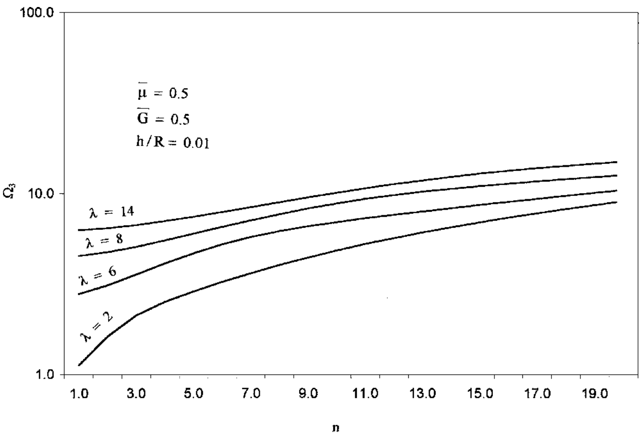

Fig. 5 Lowest eigenfrequency vs circumferential wave number for various axial wave parameters (shell on a Pasternak foundation).

Ω are plotted against longitudinal wave parameter λ in Fig. 2. If we compare the shape of the characteristics with those given in Markus[1] for an isotropic shell ($n = 1$), we notice that the shape of the largest root does not change, whereas the shape of the curve for the intermediate root gets modified as a result of the effect of the Pasternak foundation. The curve for the lowest-frequency parameter changes drastically and becomes a straight line. Change in the ratio of Young's moduli (E_1/E_2) has a significant effect on the three roots of the frequency parameters, as shown by Ω vs λ curves for $n = 2$ in Fig. 3. Such changes are also noticed in the Ω vs λ curves for $n = 0$ and 1. As we increase the value of E_2, the three frequency parameters do increase in the certain range of λ.

Again, typical variation of the eigenfrequency of the radial mode with the circumferential wave number n and axial wave parameter λ is shown in Figs. 4 and 5. Figure 4 represents the curves for an orthotropic cylindrical shell without any elastic foundation, and the shape of the curves is similar to those of Markus[1] and Paliwal and Pandey.[2] Curves in Fig. 4 have the well-known dip for the lowest radial frequency. But this well-known dip vanishes in Fig. 5, where the orthotropic shell rests on a Pasternak foundation. In this case the minimum frequency occurs at $n = 0$.

Conclusions

The effects of Pasternak foundation parameters, foundation modulus, and shear modulus, on the eigenfrequencies of orthotropic cylindrical shells are identical to those on isotropic cylindrical shells. The orthotropy of the material represented by the elastic moduli ratio has significant influence on the radial mode frequency, whereas the torsional mode frequency is only nominally influenced.

Appendix: Coefficients of Governing Equations

$$S_1 = \left[\frac{\nu_{21}}{R} + \frac{G_{12}(1-\nu_{12}\nu_{21})}{E_1 R}\right], \qquad S_2 = \frac{G_{12}(1-\nu_{12}\nu_{21})}{E_1 R^2}$$

$$S_3 = \frac{\nu_{21}}{R}, \qquad S_4 = \frac{\rho(1-\nu_{12}\nu_{21})}{E_1}$$

$$S_5 = \left[\frac{G_{12}(1-\nu_{12}\nu_{21})}{E_2} + \frac{G_{12}h^2(1-\nu_{12}\nu_{21})}{6E_2R^2}\right]$$

$$S_6 = \left(\frac{1}{R^2} + \frac{h^2}{12R^4}\right), \qquad S_7 = \left[\frac{\nu_{12}}{R} + \frac{G_{12}(1-\nu_{12}\nu_{21})}{E_2 R}\right]$$

$$S_8 = \left(\frac{h^2}{12R^4}\right), \qquad S_9 = \left[\frac{\nu_{12}h^2}{12R^2} + \frac{G_{12}h^2(1-\nu_{12}\nu_{21})}{6E_2R^2}\right]$$

$$S_{10} = \left(\frac{1}{R^2}\right), \qquad S_{11} = \frac{\rho(1-\nu_{12}\nu_{21})}{E_2}, \qquad S_{12} = -\frac{E_1h^2}{12E_2}$$

$$S_{13} = \left[\frac{E_1h^2\nu_{21}}{12R^2E_2} + \frac{G_{12}h^2(1-\nu_{12}\nu_{21})}{3R^2E_2} + \frac{h^2\nu_{21}}{12R^2}\right]$$

$$S_{14} = \left(\frac{h^2}{12R^4}\right), \qquad S_{15} = \left[\frac{E_1h^2\nu_{21}}{12R^2E_2} + \frac{G_{12}h^2(1-\nu_{12}\nu_{21})}{3R^2E_2}\right]$$

$$S_{16} = \left(\frac{h^2}{12R^4}\right), \qquad S_{17} = \left(\frac{1}{R^2}\right), \qquad S_{18} = \frac{\nu_{12}}{R}$$

$$S_{19} = \left[\frac{1}{R^2} + \frac{K(1-\nu_{12}\nu_{21})}{E_2h}\right], \qquad S_{20} = \frac{G(1-\nu_{12}\nu_{21})}{E_2h}$$

$$S_{21} = \frac{G(1-\nu_{12}\nu_{21})}{E_2hR^2}, \qquad S_{22} = \frac{\rho(1-\nu_{12}\nu_{21})}{E_2}$$

$$A_1 = \lambda^2 + n^2\overline{G_{12}}, \qquad A_2 = n\lambda(\nu_{21} + \overline{G_{12}}), \qquad A_3 = -\lambda\nu_{21}$$

$$A_4 = n\lambda(\nu_{12} + \alpha\overline{G_{12}}), \qquad A_5 = \lambda^2(1+2k)\alpha\overline{G_{12}} + n^2(1+k)$$

$$A_6 = n^3k + n\lambda^2(\nu_{12}k + 2\alpha k\overline{G_{12}}) + n, \qquad A_7 = -\lambda\nu_{12}$$

$$A_8 = n\lambda^2(\nu_{21}\alpha k + 4\alpha k\overline{G_{12}}) + n^3k + n$$

$$A_9 = \lambda^4\alpha k + n^2\lambda^2(\nu_{21}\alpha k + 4\alpha k\overline{G_{12}} + \nu_{12}k) + n^4k + (1+\alpha\bar{\mu}) + \lambda^2\alpha\bar{G} + n^2\alpha\bar{G}$$

References

[1] Markus, S., *The Mechanics of Vibrations of Cylindrical Shells*, Elsevier, Amsterdam, 1988, pp. 1–101.

[2] Paliwal, D. N., and Pandey, R. K., "The Free Vibrations of a Cylindrical Shell on an Elastic Foundation," *Journal of Vibration and Acoustics*, Vol. 120, No. 1, 1998, pp. 63–71.

[3] Paliwal, D. N., and Singh, S., "Free Vibrations of Orthotropic Cylindrical Shell on Elastic Foundation," *AIAA Journal*, Vol. 37, No. 9, 1999, pp. 1135–1139.

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